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DERIVATIONS IN MATRIX SUBRINGS

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This note is an abstract of the author's papers [1], [2] and [3]. Let R be a ring with identity and let $M_n(R)$ denotes the ring of $n \times n$ matrices over R. We say that a subring P of $M_n(R)$ is special with the relation ω if P is of the form

$$P = \{ A \in M_n(R);\ A_{ij} = 0, \text{ for } (i,j) \notin \omega \},$$

where ω is a relation (reflexive and transitive) on the set $\{1,\dots,n\}$.

We describe in this note all derivations, R-derivations and higher R-derivations of the ring P.

I. DERIVATIONS AND HIGHER DERIVATIONS IN A RING

1. Derivations. Let P be a ring with identity. An additive mapping $d: P \longrightarrow P$ is called a derivation (or an ordinary derivation) of P if $d(xy) = d(x)y + xd(y)$, for any $x,y \in P$. We denote by D(P) the set of all derivations of P. If d and d' are derivations of P then the mapping $d + d'$ is also a derivation of P , so D(P) is an abelian group.

Let $a \in P$ and let $d_a: P \longrightarrow P$ be a mapping defined by $d_a(x) = ax - xa$, for any $x \in P$. Then d_a is a derivation of P.

Let $d \in D(P)$. If there exists an element $a \in P$ such that $d = d_a$ then d is called an inner derivation (with respect to a) of P. We denote by ID(P) the est of all inner derivations of P. ID(P) is a (normal) subgroup of D(P).

We shall say that P is an NS-ring if ID(P) = D(P), that is, a ring P is an NS-ring if and only if every derivation of P is inner.

2. Higher derivations. Let P be a ring with identity and let S be a segment of $N = \{0,1,\dots\}$, that is, $S = N$ or $S = \{0,1,\dots,s\}$ for some $s \geqq 0$. A family $d = (d_m)_{m \in S}$ of mappings $d_m: P \dashrightarrow P$ is called a derivation of order s of P (where $s = \sup(S) \leqq \infty$) if the following properties are satisfied:

(1) $d_m(x + y) = d_m(x) + d_m(y)$,

(2) $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y)$,

(3) $d_0 = id_P$,

for any $x,y \in P$, $m \in S$.

The set of derivations of order s of P, denoted by $D_s(P)$, is the group under the multiplication $*$ defined by the formula

$$(d * d')_m = \sum_{i+j=m} d_i \circ d'_j ,$$

where $d,d' \in D_s(P)$ and $m \in S$.

Let $\delta: P \dashrightarrow P$ be a mapping. Then δ is an ordinary derivation of P if and only if (id_P, δ) is a derivation of order 1 of P. Therefore we may identify: $D(P) = D_1(P)$.

3. Examples of higher derivations. Let P and S be as in Section 2.

Example 3.1. Let $a \in P$, $d_0 = id_P$, and $d_m(x) = a^m x - a^{m-1}xa$, for $m \geqq 1$, $x \in P$. Then $d = (d_m)_{m \in S}$ belongs to $D_s(P)$.

Example 3.2. Let $d \in D_s(P)$, $k \in S - \{0\}$ and let $\delta = (\delta_m)_{m \in S}$ be the family of mappings from P to P defined by

$$\delta = \begin{cases} 0, & \text{if } k \nmid m \\ d_r, & \text{if } m = rk. \end{cases}$$

Then $\delta \in D_s(P)$.

The derivation d (of order s) from Example 3.1 will be denoted by $[a,1]$. The derivation δ (of order s) from Example 3.2, for $d = [a,1]$

will be denoted by $[a,k]$.

4. Inner derivations. Let P and S be as in Section 2 and let $\underline{a} = (a_m)$ (where $m \in S$) be a sequence in P. Denote by $\Delta(\underline{a})$ the element in $D_s(P)$ defined by

$$\Delta(\underline{a})_m = ([a_1,1] * [a_2,2] * \dots * [a_m,m])_m,$$

for any $m \in S$.

Definition 4.1. Let $d \in D_s(P)$. If there exists a sequence $\underline{a} = (a_m)$ of elements of P such that $d = \Delta(\underline{a})$ then d is called an inner derivation of order s of P.

Denote by $ID_s(P)$ the set of inner derivations of order s of P.

Proposition 4.2. $ID_s(P)$ is a normal subgroup of $D_s(P)$.

Proposition 4.3. The following properties are equivalent

(1) P is an NS-ring,

(2) $ID_s(P) = D_s(P)$, for any $0 < s \leqq \infty$,

(3) $ID_s(P) = D_s(P)$, for some $0 < s \leqq \infty$.

5. R-derivations. Let $R \subset P$ be rings with identity and let S be a segment of N. If a derivation (of order s) $d \in D_s(P)$ satisfies the condition

$$d_m(r) = 0,$$

for all $m \in S - \{0\}$, $r \in R$, then d is called R-derivation of order s of P, and the set of all such derivations is denoted by $D_s^R(P)$.

We define similary an ordinary R-derivation, an inner R-derivation, an inner R-derivation of order s and also, we define similary the groups $D^R(P)$, $ID^R(P)$ and $ID_s^R(P)$.

The group $D_s^R(P)$ is a subgroup of $D_s(P)$ and the group $ID_s^R(P)$ is a normal subgroup of $D_s^R(P)$.

We shall say that P is an NS-ring over R if $ID^R(P) = D^R(P)$.

Proposition 5.1. The following properties are equivalent

(1) P is an NS-ring over R,

(2) $ID_s^R(P) = D_s^R(P)$, for any $0 < s \leqq \infty$,

(3) $ID_s^R(P) = D_s^R(P)$, for some $0 < s \leqq \infty$.

II. SPECIAL SUBRINGS OF MATRIX RINGS

6. Notices. Let R be a ring with identity, n a fixed natural number and ω a reflexive and transitive relation on the set $I_n = \{1,\dots,n\}$. We denote by $M_n(R)$ the ring of $n \times n$ matrices over R and by Z(R) the center of R. Moreover, we use the following conventios:

$F(R)$ = the set of mappings from R to R,

$\bar{\omega}$ = the smallest equivalence relation on I_n containing ω,

T_ω = a fixed set of representatives of equivalence classes of $\bar{\omega}$,

A_{ij} = ij-coefficient of a matrix A,

E^{ij} = the element of the standard basis of $M_n(R)$,

$M_n(R)_\omega = \{A \in M_n(R);\ A_{ij} = 0, \text{ for } (i,j) \notin \omega\}$.

The set $P = M_n(R)_\omega$ is a subring of $M_n(R)$ called a special subring with the relation ω. Every special subring contains the ring R (via injection $r \longmapsto \bar{r}$, where $\bar{r}$ is the diagonal matrix whose all coefficients on the diagonal are equal to $r \in R$).

7. Transitive mappings and regular relations. Let G be an abelian group. A mapping $f\colon \omega \longrightarrow G$ will be called transitive iff

$$f(a,c) = f(a,b) + f(b,c),$$

for any $a\omega b$ and $b\omega c$.

If $f\colon \omega \longrightarrow G$ is a transitive mapping then we denote by [f,_] (in the case $G = R$) the mapping from ω to $F(R)$ defined by

$$[f,\_](a,b)(r) = f(a,b)r - rf(a,b),$$

for $a\omega b$ and $r \in R$. Clearly, $[f,\_]$ is transitive too.

We shall say that f is <u>trivial</u> if there exists a mapping $\sigma: I_n \dashrightarrow G$ such that

$$f(a,b) = \sigma(a) - \sigma(b),$$

for any $a\omega b$. Moreover, we shall say that f is <u>quasi-trivial</u> (in the case $G = R$) if $[f,\_]$ is trivial.

Every trivial transitive mapping from ω to R is quasi-trivial, but the converse is not necessarily true.

<u>Proposition 7.1</u>. Let $f: \omega \dashrightarrow R$ be a quasi-trivial transitive mapping. Then there exists a unique mapping $\tau: I_n \dashrightarrow F(R)$ such that

(1) $[f,\_](i,j) = \tau(i) - \tau(j)$, for all $i\omega j$,

(2) $\tau(t) = 0$, for $t \in T_\omega$.

Moreover, $\tau(1),\ldots,\tau(n)$ are inner derivations of R.

<u>Definition 7.2</u>. The relation ω is called <u>regular over an abelian group G</u> if every transitive mapping from ω to G is trivial.

8. <u>The graph $\Gamma(\omega)$ and homology groups</u>. Let $\div$ be the equivalence relation on I_n defined by: $x \div y$ iff $x\omega y$ and $y\omega x$. Denote by $[x]$ the equivalence class of $x \in I_n$ with respect to $\div$ and let I'_n be the set of all equivalence classes. We define a relation ω' of partial order on I'_n as follows:

$$[x]\omega'[y] \text{ iff } x\omega y.$$

We will denote the pair (I'_n,ω') by $\Gamma(\omega)$ and calle it the <u>graph of ω</u>. Elements of I'_n we calle <u>vertices</u> of $\Gamma(\omega)$ and pairs (a,b), where $a\omega'b$ and $a \neq b$, <u>arrows</u> of $\Gamma(\omega)$.

Let us imbed the set of the vertices of $\Gamma(\omega)$ in an Euclidean space of a sufficiently high dimension so that the vertices will be linearly independent.

If $a_0, a_1, \ldots, a_k$ are elements of I'_n such that $a_i \omega' a_{i+1}$ and $a_i \neq a_{i+1}$ for $i=0,1,\ldots,k-1$, then by $(a_0, a_1, \ldots, a_k)$ we denote the k-dimensional simplex with vertices $a_0, \ldots, a_k$. The union of all 0,1,2 or 3-dimensional such simplicies we will denote also by $\Gamma(\omega)$. Therefore, $\Gamma(\omega)$ is a simplicial complex of dimension $\leqq 3$.

Let $C_k(\omega)$, for k=0,1,2,3, be the free abelian group whose free generators are k-dimensional simplicies of $\Gamma(\omega)$. We have the following standard complex of abelian groups:

$$0 \longrightarrow C_3(\omega) \xrightarrow{\partial_3} C_2(\omega) \xrightarrow{\partial_2} C_1(\omega) \xrightarrow{\partial_1} C_0(\omega) \longrightarrow 0$$

where

$$\partial_1(a,b) = (b) - (a),$$

$$\partial_2(a,b,c) = (b,c) - (a,c) + (a,b),$$

$$\partial_3(a,b,c,d) = (b,c,d) - (a,c,d) + (a,b,d) - (a,b,c).$$

Then $H_1(\Gamma(\omega)) = \mathrm{Ker}\partial_1/\mathrm{Im}\partial_2$, $H_2(\Gamma(\omega)) = \mathrm{Ker}\partial_2/\mathrm{Im}\partial_3$ and (by the Kunneth formulas)

$$H^1(\Gamma(\omega),G) = \mathrm{Hom}(H_1(\Gamma(\omega)),G),$$

for an arbitrary abelian group G.

III DERIVATIONS IN SPECIAL SUBRINGS

9. Examples of derivations. Let $P = M_n(R)$ be a special subring of $M_n(R)$.

Example 9.1. Assume that $f\colon \omega \longrightarrow R$ is a quasi-trivial transitive mapping and denote by Δ^f the mapping from P to P defined by

$$\Delta^f(B)_{pq} = B_{pq}f(p,q) + \tau_f(p)(B_{pq}),$$

for $B \in P$, $p\omega q$, where τ_f is the mapping τ from Proposition 7.1. Then Δ^f is a derivation of P. Moreover Δ^f is inner if and only if f is trivial.

Example 9.2. Let $\delta = \{\delta_t;\ t \in T_\omega\}$ be a set of derivations of R. Denote by Θ_δ the mapping from P to P defined by

$$\Theta_\delta(B)_{pq} = \delta_t(B_{pq}),$$

for $B \in P, p\omega q$, where $t \in T_\omega$ such that $p\bar{\omega}t$.

Then Θ_δ is a derivation of P. Moreover, Θ_δ is inner if and only if δ_t is inner for any $t \in T_\omega$.

10. A description of D(P). Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$. The following theorem describes all derivations of P.

Theorem 10.1. Every derivation d of P has a unique representation:

$$d = d_A + \Delta^f + \Theta_\delta,$$

where (1) d_A is an inner derivation of P with respect to a matrix $A \in P$ such that $A_{pp} = 0$, for $p=1,\dots,n$,

(2) $f\colon \omega \dashrightarrow R$ is a quasi-trivial transitive mapping and Δ^f is the derivation from Example 9.1,

(3) $\delta = \{\delta_t;\ t \in T_\omega\}$ is a set of derivations of R and Θ_δ is the derivation from Example 9.2.

The next theorem describes special subrings which are NS-rings.

Theorem 10.2. The following conditions are equivalent

(1) P is an NS-ring,

(2) R is an SN-ring and the relation ω is regular over Z(R).

11. R-derivations of $M_n(R)_\omega$. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$.

Example 11.1. Let $f\colon \omega \dashrightarrow Z(R)$ be a transitive mapping and denote by Δ^f the mapping from P to P defined by $\Delta^f(B)_{pq} = f(p,q)B_{pq}$, for $B \in P$ and $p\omega q$. Then Δ^f is an R-derivation of P. Moreover Δ^f is inner if and only if f is trivial.

Theorem 11.2. Any R-derivation d of P has a unique representation

$$d = d_A + \Delta^f,$$

where (1) d_A is an inner derivation of P with respect to a matrix $A \varepsilon P$ such that $A_{ij} \varepsilon Z(R)$ for $i,j=1,\dots,n$, and $A_{ii} = o$ for $i=1,\dots,n$,

(2) $f: \omega \dashrightarrow Z(R)$ is a transitive mapping and Δ^f is the derivation from Example 11.1.

Theorem 11.3. The following conditions are equivalent

(1) P is an NS-ring over R,

(2) ω is regular over Z(R).

Corollary 11.4. If d and δ are R-derivations of R then the derivation $d\delta - \delta d$ is inner.

Corollary 11.5. If d is an R-derivation of P then $d(Z(R)) = 0$.

Corollary 11.6. If d is an R-derivation of P and U is an ideal of P then D(U)   U.

12. An example of non-inner R-derivation. For $n \leq 3$ every relation ω (reflexive and transitive) on I_n is regular over any group. Therefore (by Theorem 11.3), in this case any special subring of $M_n(R)$ has only inner R-derivations. For n=4 it is not true. Let ω_0 be the relation on $I_4 = \{1,2,3,4\}$ defined by the graph

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 3 \\ 4 & 2 \arrow[l] \arrow[u] \end{tikzcd},$$

that is, $\omega_0 = \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4),(2,3),(2,4)\}$.

Denote by $S_4(R)$ the special subring of $M_4(R)$ with the relation ω_0

Then we have

$$S_4(R) = \begin{bmatrix} R & 0 & R & R \\ 0 & R & R & R \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix} .$$

Consider the mapping $d\colon S_4(R) \dashrightarrow S_4(R)$ defined by

$$d(\begin{bmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix}) = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Then d is an R-derivation of $S_4(R)$ and d is not inner.
In [1] there is a description of the group $D^R(S_4(R))$. Note one of the properties of R-derivations of the ring $S_4(R)$.

<u>Corollary 12.1</u>. If d_1 and d_2 are R-derivations of $S_4(R)$ then the composition d_1d_2 is also R-derivation of $S_4(R)$.

13. <u>A description of regular relations</u>. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$.

We know, by Theorem 10.2, that P is an NS-ring if and only if R is an NS-ring and the relation ω is regular over $Z(R)$. We know also, by Theorem 11.3, that P is an NS-ring over R if and only if the relation ω is regular over $Z(R)$.

In this section we give some suffiecient and necessary conditions for the relation ω to be regular over an abelian group.

We may reduce our consideration to the case where ω is connected (that is, for any $a,b \in I_n$ there exist elements $a_1,\dots,a_r \in I_n$ such that $a = a_1$, $b = a_r$ and $a_i\omega a_{i+1}$ or $a_{i+1}\omega a_i$, for $i=1,\dots,r-1$), because it is easy to show the following

<u>Proposition 13.1</u>. Let G be an abelian group. The relation ω is regular over G if and only if every connected component of ω is regular over G.

The next proposition says that we may also reduce our consideration to the case where ω is a partial order.

Proposition 13.2. ω is regular over G if and only if ω'(see Section 8) is regular over G.

Now we may give a description of regular relations.

Theorem 13.3. Assume that ω is a connected partial order. The following properties are equivalent:

(1) ω is regular over some non-zero group,

(2) ω is regular over every torsion-free group,

(3) ω is regular over some torsion-free group,

(4) ω is regular over Z,

(5) $H_1(\Gamma(\omega))$ is finite,

(6) $H^1(\Gamma(\omega),G) = 0$, for any torsion-free group G.

Theorem 13.4. Assume that ω is connected partial order. The following properties are equivalent:

(1) ω is regular over any group,

(2) ω is regular over Q/Z,

(3) $H_1(\Gamma(\omega)) = 0$,

(4) $H^1(\Gamma(\omega),G) = 0$, for any group G.

Theorem 13.5. Assume that ω is connected partial order, such that the order of the group $H_1(\Gamma(\omega))$ is equal to $m > 1$. Let G be an abelian group. The following properties are equivalent:

(1) ω is regular over G,

(2) G is an m-torsion-free group,

(3) $H^1(\Gamma(\omega), G) = 0$.

Corollary 13.6. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$. The following properties are equivalent

(1) Every R-derivation of P is inner,

(2) The relation ω is regular over Z(R),

(3) $H^1(\Gamma(\omega),Z(R)) = 0$.

14. Examples. Let $P = M_n(R)_\omega$ where

a) $n \leqq 3$, or

b) the graph $\Gamma(\omega)$ is a tree, or

c) the graph $\Gamma(\omega)$ is a conne (that is, there exists $b \in I_n$ such that $b\omega a$ or $a\omega b$ for any $a \in I_n$), in particular $P = M_n(R)$ is the ring of $n \times n$ matrices over R, or P is the ring of triangular $n \times n$ matrices over R.

Then every R-derivation (or every drivation, if every derivation of R is inner) of P is inner.

By Theorem 13.5 it follows that there exist relations ω which are regular over some groups and which are not regular over another groups. In the paper [1] there is an example of such a relation ω (for n=17) that if R is 2-torsion-free ring then $P = M_n(R)_\omega$ is an NS-ring over R, and if char(R) = 2 then $P = M_n(R)_\omega$ is not an NS-ring over R.

IV HIGHER DERIVATIONS IN SPECIAL SUBRINGS

15. An example of higher derivations. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$, S a segment of N, and let $\underline{d} = \{d^{(t)};\ t \in T_\omega\}$ be a family of derivations of order s (where s = sup(S)) of the ring R.

Denote by $\Theta(\underline{d})$ the sequence $(d_m)_{m \in S}$ of mappings from P to P defined by

$$d_m(A)_{ij} = d_m^{(\upsilon(i))}(A_{ij}),$$

for $m \in S$, $A \in P$, where $\upsilon : I_n \longrightarrow T_\omega$ is the mapping: $\upsilon(p) = t$ iff $p\bar{\omega}t$.

Then $\Theta(\underline{d})$ is a derivation of order s of P. If $\underline{d} \neq 0$ then the derivation $\Theta(\underline{d})$ is not an R-derivation.

In the next sections of this note we shall interesting only in R-derivations of order s of P.

16. Transitive mappings of order s. A sequence $f = (f_m)_{m \in S}$ of mappings $f_m : \omega \dashrightarrow Z(R)$ is called a transitive mapping of order s (from ω to R) if the following properties are satisfied:

(1) $f_0(p,q) = 1$, for all $p\omega q$,

(2) $f_m(p,r) = \sum_{i+j=m} f_i(p,q)f_j(q,r)$, for all $m \in S$ and $p\omega q$ and $q\omega r$.

If $f = (f_m)_{m \in S}$ is a transitive mapping of order s then

$$f_1(p,r) = f_1(p,q) + f_1(q,r),$$

for any $p\omega q\omega r$ so, $f_1 : \omega \dashrightarrow Z(R)$ is a transitive mapping in the sense of Section 7.

17. R-derivations of order s. In this section we give a description of the group $D_s^R(P)$.

Example 17.1. Let $f = (f_m)_{m \in S}$ be a transitive mapping of order s from ω to Z(R). Denote by Δ^f the sequence $(\Delta_m^f)_{m \in S}$ of mappings $\Delta_m^f : P \dashrightarrow P$ defined by the following formula:

$$\Delta_m^f(A)_{pq} = f_m(p,q)A_{pq},$$

for all $A \in P$ and $p\omega q$.

Then Δ^f is an R-derivation of order s of P.

Theorem 17.2. Every R-derivation d of order s of P has a unique representation:

$$d = \Delta(\underline{A}) * \Delta^f,$$

where

(1) $\underline{A} = (A^{(m)})_{m \in S - \{0\}}$ is a sequence of matrices $A^{(m)} \in P \cap M_n(Z(R))$ such that $A^{(m)}_{ii} = 0$, for $i=1,\ldots,n$, and $\Delta(\underline{A})$ is the inner derivation of order s with respect to $\underline{A}$;

(2) f is a transitive mapping of order s from ω to R and Δ^f is the R-derivation from Example 17.1.

Corollary 17.3. If $d \in D^R_s(P)$ and U is an ideal of P then $d_m(U) \subset U$, for all $m \in S$.

Corollary 17.4. If $d \in D^R_s(P)$, then $d_m(Z(R)) = 0$, for all $m \in S - \{0\}$.

Corollary 17.5. Assume that there do not exist three defferent elements $a,b,c \in I_n$ such that $a\omega b\omega c$. Let $d = (d_m)_{m \in S}$ be a sequence of mappings from P to P such that $d_0 = id_P$. Then d is an R-derivation of order s of P if and only if every mapping d_m (for $m \in S - \{0\}$) is an ordinary R-derivation of P.

18. Integrable derivations. Let $S = \{0,1,\ldots,s\}$, where $s < \infty$. Assume that S' is a segment of N such that $S \subsetneq S'$. We say that an R-derivation $d \in D^R_s(P)$ is s'-integrable (where $s' = \sup(S') \leqq \infty$) if there exists an R-derivation $d' = (d'_m)_{m \in S'}$ of order s' of P such that $d'_m = d_m$, for all $m \in S$.

In the paper [3] there are some necessary conditions for any R-derivation of order s of P to be s'-integrable, and there is an example of non-integrable R-derivation (In this example n=17 and $R = Z_2$). In this paper there are also proofs of the following two partial results:

Theorem 18.1. Let $s < s' \leqq \beta$. If $H_2(\Gamma(\omega)) = 0$ and $H_1(\Gamma(\omega))$ is a free abelian group then every R-derivation of order s of P is s'-integrable.

Theorem 18.2. Assume that the homology group $H_1(\Gamma(\omega))$ is free abelian. Then

(1) Every R-derivation of order $s < 3$ of P is 3-integrable.

(2) If R is 2-torsion-free then every R-derivation of order < 5 of P is 5-integrable.

(3) If R is 6-torsion-free then every R-derivation of order < 7 of P is 7-integrable.

References

[1] A.Nowicki, Derivations of special subrings of matrix rings and regular graphs, Tsukuba J.Math.,7(1983),281-297.

[2] ---, Inner derivations of higher order, Tsukuba J.Math., 8(1984), 219-225.

[3] ---, Higher R-derivations of special subrings of matrix rings, Tsukuba J.Math.,8(1984),227-253.

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